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CONTENTS

Prime ideal of N-group

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Abstract

In this paper we introduce the notion of prime ideal of an N-group Γ. We obtain some results for a prime ideal of an N-group Γ. Also we show that if Q is a completely prime ideal of N and S is a complement of Q in N then $M = \{\frac{q}{s} | q \in Q$ and $s \notin Q\}$ is a unique maximal ideal in a near-ring of right quotients of N with respect to S.

Key words: prime ideal, maximal ideal, N-group.

2010 Mathematics Subject Classification: 16Y30, 16Y60.

1 Introduction

Throughout this paper N denotes a zero-symmetric near-ring with identity and Γ denotes an N-group. If N has a unit element 1, and if $1.\gamma = \gamma$ for every element $\gamma \in \Gamma$, then Γ is called a unital N-group. In this paper, we assume that all N-groups are unital. So far only ideal is defined in a N-group Γ. In this paper we introduce the notion of prime ideal of an N-group Γ. We obtain some results for a prime ideal of an N-group Γ and maximal ideal of right quotients.

2 Preliminaries

In this section, we recall the definitions needed for our purpose.

Definition 2.1 *An ideal P of N is called completely prime if for any $a, b \in N$, $ab \in P$ implies either $a \in P$ or $b \in P$.*

Definition 2.2 *An ideal Q of N is called completely semiprime if for any $a \in N$, $a^2 \in Q$ implies $a \in Q$.*

Remark 2.3

1. If P is a completely prime ideal of N, then P is a prime ideal of N.
2. If Q is a completely semiprime ideal of N, then Q is a semiprime ideal of N.

Lemma 2.4 *Let I be a completely semiprime ideal of N. Then for any $a, b \in N$, $ab \in I$ implies $ba \in I$ and $aNb \subseteq I$.*

Proof. Assume that $ab \in I$ for any $a, b \in N$. Now $(ba)^2 = b(ab)a \in I$. Since I is completely semiprime, we have $ba \in I$. Let $n \in N$. Now $(anb)^2 = an(ba)nb \in I$. Since I is completely semiprime, $anb \in I$. Hence $aNb \subseteq I$. ■

3 Main Results

In this section, we obtain some results for a prime ideal of an N-group Γ and maximal ideal of right quotients.

Definition 3.1 *An ideal P of Γ is said to be prime if $P \neq \Gamma$ and whenever $n\gamma \in P$ (where $n \in N$ and $\gamma \in \Gamma$) then $n \in (P : \Gamma)$ or $\gamma \in P$.*

If P is prime, then the ideal $p = (P : \Gamma)$ is a prime ideal of N, and P is said to be p-prime.

Lemma 3.2 *If P is a prime ideal of Γ, then $(P : \Gamma)$ is a completely semiprime ideal of N.*

Proof. Assume that P is a prime ideal of Γ. Suppose that $a \in N$ such that $a^2 \in (P : \Gamma)$. Let $\gamma \in \Gamma$. Then $a(a\gamma) \in P$. Since P is a prime ideal of Γ, we have $a \in (P : \Gamma)$ or $a\gamma \in P$ for all $\gamma \in \Gamma$. Thus, in any case, $a \in (P : \Gamma)$. Hence $(P : \Gamma)$ is a completely semiprime ideal of N. ■

Lemma 3.3 *If* P *is a prime ideal of* Γ, *then* $(P:\Gamma)$ *is a completely prime ideal of* N.

Proof. Assume that P is a prime ideal of Γ. Let $a, b \in N$ such that $ab \in (P:\Gamma)$. By Lemma 3.2, $(P:\Gamma)$ is a completely semiprime ideal of N. Then $\langle a \rangle \langle b \rangle \subseteq (P:\Gamma)$. Since $(P:\Gamma)$ is a prime ideal of N, we have $a \in (P:\Gamma)$ or $b \in (P:\Gamma)$. Hence $(P:\Gamma)$ is a completely prime ideal of N. ■

Hereafter we assume that every N-subgroup is an ideal and $n(\gamma_1+\gamma_2) = n\gamma_1 + n\gamma_2$ for all $n \in N$, $\gamma_1, \gamma_2 \in \Gamma$.

Theorem 3.4 *Let* N *be a near-ring and* I *an ideal which is contained in every maximal ideal. Then for any* $i \in I$, $1+i$ *and* $i+1$ *have left inverses.*

Proof. Let M be a maximal ideal of N. Let $i \in I$. Suppose $N(1+i) \subseteq M$. Since $1+i \in N(1+i)$, we have $1+i \in M$. Since $-i \in M$, we have $1 \in M$, a contradiction. Hence $N(1+i) = N$. Since $1 \in N(1+i)$, we can find $x \in N$ such that $1 = x(1+i)$. Similarly $i+1$ has left inverse. ■

We have the following corollary from Theorem 3.4.

Corollary 3.5 *[2] Let* R *be a commutative ring and* I *an ideal which is contained in every maximal ideal. Then for any* $i \in I$, $1+i$ *is a unit.*

Theorem 3.6 *Let* N *be a near-ring and* I *an ideal which is contained in every maximal ideal. Let* Γ *be a finitely generated* N*-group. Suppose that* $I\Gamma = \Gamma$, *then* $\Gamma = 0$.

Proof. We use induction on the number of generators. Let γ_1 be the generator. Since $I\Gamma = \Gamma$, there exists $i \in I$ and $\gamma \in \Gamma$ such that $\gamma_1 = i\gamma$. Since γ_1 is the generator, we can find $n \in N$ such that $\gamma = n\gamma_1$. Now $\gamma_1 = in\gamma_1$. Then $(1+(-in))\gamma_1 = 0$. By Theorem 3.4, there exists $x \in N$ such that $x(1+(-in)) = 1$. Therefore, $\gamma_1 = 0$. Assume that the result is true for $n-1$ generators. Let $\gamma_1, \gamma_2, ..., \gamma_n$ be the generators. Since $I\Gamma = \Gamma$, there exists $i \in I$ and $\gamma \in \Gamma$ such that $\gamma_1 = i\gamma$. Since $\gamma_1, \gamma_2, ..., \gamma_n$ are the generators, we can find $a_1, a_2, ..., a_n \in N$ such that $\gamma = a_1\gamma_1 + a_2\gamma_2 + ... + a_n\gamma_n$. Now $\gamma_1 = ia_1\gamma_1 + ia_2\gamma_2 + ... + ia_n\gamma_n$. Thus, $(-ia_1+1)\gamma_1 = ia_2\gamma_2 + ... + ia_n\gamma_n$. By Theorem 3.4, there exists $x \in N$ such that $x(-ia_1+1) = 1$. Hence $\Gamma = 0$. ■

Notation 3.7 Let S be a subsemigroup of $(N, .)$. A near-ring of right quotients of N with respect to S is denoted by N_S.

Lemma 3.8 *Let* N *be a near-ring and* Q *a completely prime ideal of* N. *Let* S *be the complement of* Q *in* N. *Then* $M = \{\frac{q}{s} | q \in Q \text{ and } s \notin Q\}$ *is a unique maximal ideal in a near-ring of right quotients of* N *with respect to* S.

Proof. Let $\frac{q_1}{s_1}, \frac{q_2}{s_2} \in M$. Now $\frac{q_1}{s_1} - \frac{q_2}{s_2} = \frac{q_1 s_3 - q_2 n_3}{s_1 s_3} \in M$ where $n_3 \in N, s_3 \in S$ fulfills $s_2 n_3 = s_1 s_3 \in S$. Let $\frac{n}{s} \in N_S$ and $\frac{q}{s'} \in M$. Now $\frac{n}{s} + \frac{q}{s'} - \frac{n}{s} = \frac{ns_1 + qn_1}{ss_1} - \frac{n}{s}$ where $n_1 \in N, s_1 \in S$ is such that $s' n_1 = ss_1 \in S$. Then $\frac{n}{s} + \frac{q}{s'} - \frac{n}{s} = \frac{(ns_1 + qn_1)s_2 - nn_2}{ss_1 s_2}$ where $n_2 \in N, s_2 \in S$ satisfies $sn_2 = ss_1 s_2 \in S$. Thus, $\frac{n}{s} + \frac{q}{s'} - \frac{n}{s} = \frac{ns_1 s_2 + qn_1 s_2 - ns_1 s_2}{ss_1 s_2} \in M$. Let $\frac{n}{s} \in N_S$ and $\frac{q}{s_1} \in M$. Now $\frac{q}{s_1}\frac{n}{s} = \frac{qn_2}{ss_2} \in M$ where $n_2 \in N, s_2 \in S$ with $qs_2 = ns_2 \in S$. Let $\frac{n'}{s'}, \frac{n''}{s''} \in N_S$ and $\frac{q}{s} \in M$. Now $\frac{n'}{s'}(\frac{n''}{s''} + \frac{q}{s}) - (\frac{n'}{s'}\frac{n''}{s''}) = \frac{n'}{s'}(\frac{n'' s_1 + qn_1}{s'' s_1}) - (\frac{n'}{s'}\frac{n''}{s''})$ where $n_1 \in N, s_1 \in S$ satisfies $sn_1 = s'' s_1 \in S$. Then $\frac{n'}{s'}(\frac{n''}{s''} + \frac{q}{s}) - (\frac{n'}{s'}\frac{n''}{s''}) = \frac{n'}{s'}(\frac{n'' s_1 + qn_1}{s'' s_1}) - (\frac{n' q'}{s'' s_2})$ where $q' \in Q, s_2 \in S$ is such that $n' s_2 = n'' s_2 \in S$. Hence $\frac{n'}{s'}(\frac{n''}{s''} + \frac{q}{s}) - (\frac{n'}{s'}\frac{n''}{s''}) = \frac{n' q''}{s'' s_1 s_3} - \frac{n' q'}{s'' s_2} \in M$ where $q'' \in Q, s_3 \in S$ with $n' s_3 = (n'' s_1 + qn_1)s_3 \in S$.

Suppose that M is not maximal, say $M \subset M_1$ for some maximal ideal M_1 of N_S. Let $\frac{a}{s} \in M_1$ where $a \notin Q$, $s \in S$. Now $\frac{a}{s}\frac{s}{a} = \frac{as_1}{as_1}$ where $s_1 \in S$ fulfills $as_1 = ss_1 \in S$. Then $M_1 = N_S$, a contradiction. Hence M is a unique maximal ideal in a near-ring of right quotients of N with respect to S. ∎

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